Ehrhart polynomials of polytopes and orthogonal polynomial systems (Designs, Codes, Graphs and Related Areas)

Higashitani, Akihiro

数理解析研究所講究録
第 1889 巻 2014 年 74-83

Ehrhart polynomials of polytopes and orthogonal polynomial systems

Akihiro Higashitani[1]
Osaka University

Abstract

In this draft, for the study of the zeros of the Ehrhart polynomials of reflexive polytoeps, we consider a relation between the Ehrhart polynomials of reflexive polytopes and orthogonal polynomial systems.

1 Introduction

1.1 Ehrhart polynomials of integral convex polytopes

Let $\mathcal{P} \subset \mathbb{R}^N$ be an integral convex polytope, which is a convex polytope all of whose vertices have integer coordinates, of dimension n. Given a positive integer $x \in \mathbb{Z}_{>0}$, we write

$$i(\mathcal{P}, x) = |x\mathcal{P} \cap \mathbb{Z}^N|,$$

where $x\mathcal{P} = \{x\alpha : \alpha \in \mathcal{P}\}$ and $|\cdot|$ denotes the cardinality. The studies on $i(\mathcal{P}, x)$ originated in the work of Ehrhart ([9]), who proved that the enumerative function $i(\mathcal{P}, x)$ can be described as a polynomial in x of degree n whose constant term is 1. We call the polynomial $i(\mathcal{P}, x)$ the *Ehrhart polynomial* of $\mathcal{P}$. We refer the reader to [5, Chapter 3] or [12, Part II] for the introduction to the theory of Ehrhart polynomials.

We also define the integers $\delta_0, \delta_1, \ldots$ by the following formula

$$\sum_{x=0}^{\infty} i(\mathcal{P}, x)t^x = \frac{\sum_{i=0}^{\infty} \delta_i t^i}{(1-t)^{n+1}}.$$

Since $i(\mathcal{P}, x)$ is a polynomial in x of degree n, we know that $\delta_i = 0$ for every $i > n$ (consult, e.g., [18, Corollary 4.3.1]). The integer sequence $\delta(\mathcal{P}) = (\delta_0, \delta_1, \ldots, \delta_n)$ is

[1]E-mail:a-higashitani@cr.math.sci.osaka-u.ac.jp

called the *δ-vector* (alternately, h^*-vector or Ehrhart h-vector) of $\mathcal{P}$. The following properties on δ-vectors are well known:

- One has $\delta_0 = 1$, $\delta_1 = |\mathcal{P} \cap \mathbb{Z}^N| - (n+1)$.
- One has $\delta_n = |(\mathcal{P} \setminus \partial\mathcal{P}) \cap \mathbb{Z}^N|$. Hence, we also have $\delta_1 \geq \delta_n$.
- Each δ_i is nonnegative ([17]).
- The leading coefficient of $i(\mathcal{P}, x)$, which equals $\sum_{i=0}^{n} \delta_i / n!$, coincides with the volume of $\mathcal{P}$ ([18, Corollary 3.20]).
- The Ehrhart polynomial can be described like
$$i(\mathcal{P}, x) = \sum_{k=0}^{n} \delta_k \binom{x+n-k}{n}.$$

1.2 Reflexive polytopes

For an integral convex polytope $\mathcal{P} \subset \mathbb{R}^n$ of dimension n, we say that $\mathcal{P}$ is a *reflexive polytope* if $\mathcal{P}$ contains the origin of $\mathbb{R}^n$ as the unique interior integer point and the dual polytope $\mathcal{P}^\vee$ of $\mathcal{P}$ is also integral, where $\mathcal{P}^\vee = \{x \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } y \in \mathcal{P}\}$ and $\langle \ , \ \rangle$ denotes the usual inner product of $\mathbb{R}^n$.

Recently, the zeros of the Ehrhart polynomials of integral convex polytopes have been studied by many researchers ([4, 6, 7, 10, 11, 14, 15]). Especially, the distribution of the real parts of the zeros is of particular interest. In [4, Conjecture 1.4], it was conjectured that all the zeros α of the Ehrhart polynomial of an integral convex polytope of dimension n satisfy $-n \leq \Re(\alpha) \leq n-1$, where $\Re(\alpha)$ stands for the real part of α. However, this conjecture has been disproved by [11] and [15].

On the other hand, for a reflexive polytope $\mathcal{P}$ of dimension n, its Ehrhart polynomial has an extremal property. More precisely, the following functional equation holds:
$$i(\mathcal{P}, x) = (-1)^n i(\mathcal{P}, -x-1).$$
This says that all the zeros of the Ehrhart polynomials of reflexive polytopes are distributed symmetrically in the complex plane with respect to the vertical line $\Re(z) = -1/2$. Note that the line $\Re(z) = -1/2$ is the bisector of the vertical strip $-n \leq \Re(z) \leq n-1$. Hence the problem of which reflexive polytope whose Ehrhart polynomial has the property

all the zeros of the Ehrhart polynomial have the same real part $-1/2$ $\cdots\cdots$ ($\sharp$)

arises naturally and looks fascinating. This is solved by [6, Proposition 1.9] in the case of $n \leq 4$. In order to try this problem for the general case, we employ the idea of orthogonal polynomials.

1.3 Orthogonal polynomial and its zeros

We refer the reader to [8] for the introduction to orthogonal polynomial systems. Let $\{f_n(x)\}_{n=0}^{\infty}$ be an orthogonal polynomial system with respect to a positive-definite moment functional. (In the rest of this draft, we often write "a (positive-definite) OPS" instead of an orthogonal polynomial system with respect to a (positive-definite) moment functional.) We say that a polynomial is a (positive-definite) orthogonal polynomial if it is one polynomial of some (positive-definite) OPS. On the zeros of an orthogonal polynomial, the following is a well-known fact:

Theorem 1 (cf. [8, Theorem 5.2]) *The zeros of $f_n(x)$ are all real and simple.*

On the other hand, for the Ehrhart polynomial $i(\mathcal{P}, x)$ of some reflexive polytope $\mathcal{P}$ of dimension n, let $f_n(x) = i(\mathcal{P}, \sqrt{-1}x - 1/2)$. If we know that $f_n(x)$ is a positive-definite OPS, then all the zeros of $f_n(x)$ are real numbers by Theorem 1. It then follows from $f_n(x) = i(\mathcal{P}, \sqrt{-1}x - 1/2)$ that $\mathcal{P}$ has the property ($\natural$), that is, all the zeros of $i(\mathcal{P}, x)$ have the same real part $-1/2$.

Such a consideration would naturally lead the author into the temptation to study the following problem:

Problem 2 *Find or characterize reflexive polytopes $\mathcal{P}$ whose Ehrhart polynomial $i(\mathcal{P}, x)$ satisfies that $i(\mathcal{P}, \sqrt{-1}x - 1/2)$ is a positive-definite orthogonal polynomial.*

A challenge to this problem is significant towards a complete characterization of reflexive polytopes which have the property ($\natural$).

1.4 Organization

A brief organization of this draft is as follows. In Section 2, we discuss a relation between the Ehrhart polynomials of reflexive polytopes and OPS. Especially, we consider a certain three-terms recurrence formula for the Ehrhart polynomials of reflexive polytopes (Proposition 4). In Section 3, we find four examples of reflexive polytopes each of whose Ehrhart polynomials $i(\mathcal{P}, x)$ satisfies that $i(\mathcal{P}, \sqrt{-1}x - 1/2)$ is a positive-definite orthogonal polynomial (Examples 5, 6, 7 and 8). Finally, in Section 4, as one small partial answer for Problem 2, we present Theorem 12.

2 Ehrhart polynomials of reflexive polytopes and the three-terms recurrence formula

In this section, we study a relation between the Ehrhart polynomials of reflexive polytopes and OPS.

First, we recall the following proposition, which gives a characterization of reflexive polytopes in terms of Ehrhart polynomials or δ-vectors.

Proposition 3 (cf. [3, 13]) *Let $\mathcal{P}$ be an integral convex polytope of dimension n, $i(\mathcal{P},x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + 1$ its Ehrhart polynomial and $\delta(\mathcal{P}) = (\delta_0, \delta_1, \ldots, \delta_n)$ its δ-vector. Then the following four conditions are equivalent:*

(a) *$\mathcal{P}$ is unimodularly equivalent to a reflexive polytope;*

(b) *$\delta(\mathcal{P}_n)$ is symmetric, i.e., $\delta_j = \delta_{n-j}$ for every $0 \leq j \leq n$;*

(c) *the functional equation $i(\mathcal{P},x) = (-1)^n i(\mathcal{P},-x-1)$ holds;*

(d) $na_n = 2a_{n-1}$.

Next, we discuss when a sequence of the Ehrhart polynomials of reflexive polytopes forms an OPS.

Proposition 4 *Let $\mathcal{P}_n$, $n \geq 0$, be reflexive polytopes of dimension n and let $f_n(x) = i(\mathcal{P}_n, x)$. Then the sequence of the Ehrhart polynomials $\{f_n(x)\}_{n=0}^{\infty}$ is an OPS if and only if $\{f_n(x)\}_{n=0}^{\infty}$ satisfies the three-terms recurrence formula*

$$f_n(x) = M_n(2x+1)f_{n-1}(x) + (1 - M_n)f_{n-2}(x) \quad \text{for } n \geq 2, \tag{1}$$

where each M_n is a positive rational number. Moreover, let $g_n(x) = f_n(\sqrt{-1}x - 1/2)/k_n$, where k_n is the leading coefficient of the polynomial $f_n(\sqrt{-1}x - 1/2)$. Then $\{g_n(x)\}_{n=0}^{\infty}$ is a positive-definite OPS if and only if $\{g_n(x)\}_{n=0}^{\infty}$ satisfies the three-terms recurrence formula

$$g_n(x) = xg_{n-1}(x) - N_n g_{n-2}(x) \quad \text{for } n \geq 2,$$

where each N_n is a rational number with $N_n > 0$ for $n \geq 2$.

A sketch of proof is as follows. In general, by [8, Theorem 4.1] together with [8, Theorem 4.4], a sequence $\{h_n(x)\}_{n=0}^{\infty}$ of the polynomials $h_n(x)$ of degree n is OPS if and only if this satisfies a certain three-terms recurrence formula, which is of the form

$$h_n(x) = (A_n x + B_n)h_{n-1}(x) + C_n h_{n-2}(x).$$

Thanks to Proposition 3, we obtain that $A_n = 2B_n$ in the case of the Ehrhart polynomials of reflexive polytopes. Moreover, since the constant of the Ehrhart polynomial is always 1, we also obtain $B_n + C_n = 1$. In addition, it is also known that $\{h_n(x)\}_{n=0}^{\infty}$ is a positive-definite OPS if and only if C_n is always negative for each $n \geq 2$.

3 Examples of reflexive polytopes whose Ehrhart polynomials satisfy (1)

In this section, we present some examples of reflexive polytopes. The Ehrhart polynomials of such examples satisfy the recurrence (1).

Let $\mathbf{e}_1, \ldots, \mathbf{e}_n \in \mathbb{R}^n$ be the unit vectors of $\mathbb{R}^n$.

Example 5 (cross polytope) Let $\mathcal{P}_n = \mathrm{conv}(\{\pm\mathbf{e}_1, \ldots, \pm\mathbf{e}_n\})$. Then this is called a *cross polytope* of dimension n. Let $f_n(x) = i(\mathcal{P}_n, x)$ be its Ehrhart polynomial and $\delta(\mathcal{P}_n)$ its δ-vector. Then it is well known that $\delta(\mathcal{P}_n) = (\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n})$, i.e.,

$$f_n(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{x+n-k}{n}.$$

Note that the leading coefficient of $f_n(x)$ is equal to $\sum_{k=0}^{n} \binom{n}{k}/n! = 2^n/n!$.

Now one can check by a direct computation that $f_n(x)$ satisfies (1) with $M_n = 1/n$, that is,

$$f_n(x) = \frac{1}{n}(2x+1) f_{n-1}(x) + \frac{n-1}{n} f_{n-2}(x) \quad \text{for } n \geq 2. \tag{2}$$

Let

$$\widetilde{f_n}(x) = \frac{n! \cdot f_n\left(\sqrt{-1}x - \frac{1}{2}\right)}{\sqrt{-1}^n 2^n}.$$

Then $\widetilde{f_n}(x)$ is a monic polynomial in x. From (2), one sees that $\widetilde{f_n}(x)$ satisfies the recurrence

$$\widetilde{f_n}(x) = x\widetilde{f_{n-1}}(x) - \frac{(n-1)^2}{4} \widetilde{f_{n-2}}(x) \quad \text{for } n \geq 2.$$

Since $(n-1)^2/4 > 0$ for $n \geq 2$, this says that $\{\widetilde{f_n}(x)\}_{n=0}^{\infty}$ is a positive-definite OPS by Proposition 4. Hence $\widetilde{f_n}(x)$ has the zeros which are all real and simple.

Therefore, we conclude that each cross polytope has the property ($\sharp$).

Example 6 (dual of Stasheff polytope) Let $\mathcal{P}_n = \mathrm{conv}(\{\pm\mathbf{e}_1, \ldots, \pm\mathbf{e}_n\} \cup \{\mathbf{e}_i + \cdots + \mathbf{e}_j : 1 \leq i < j \leq n\})$. Note that this is a convex hull of the *almost positive roots* of type A Weyl group and this is a dual polytope of so-called the *Stasheff polytope* of dimension n. Then it is known by Athanasiadis [2] that $\delta(\mathcal{P}_n) = (\frac{1}{n+1}\binom{n+1}{0}\binom{n+1}{1}, \frac{1}{n+1}\binom{n+1}{1}\binom{n+1}{2}, \ldots, \frac{1}{n+1}\binom{n+1}{n}\binom{n+1}{n+1})$, i.e.,

$$f_n(x) = \sum_{k=0}^{n} \frac{1}{n+1} \binom{n+1}{k} \binom{n+1}{k+1} \binom{x+n-k}{n}.$$

Here we note that each $\frac{1}{n+1}\binom{n+1}{k}\binom{n+1}{k+1}$ is known as the *Narayana number*. We notice that the leading coefficient of $f_n(x)$ is equal to $\sum_{k=0}^{n}\frac{1}{n+1}\binom{n+1}{k}\binom{n+1}{k+1}/n! = C_{n+1}/n!$, where C_n is the *Catalan number*.

Now one can check that $f_n(x)$ satisfies (1) with $M_n = (2n+1)/n(n+2)$, that is,

$$f_n(x) = \frac{2n+1}{n(n+2)}(2x+1)f_{n-1}(x) + \frac{(n+1)(n-1)}{n(n+2)}f_{n-2}(x) \quad \text{for } n \geq 2.$$

Let

$$\widetilde{f_n}(x) = \frac{n! \cdot f_n\left(\sqrt{-1}x - \frac{1}{2}\right)}{\sqrt{-1}^n C_{n+1}}.$$

Then $\widetilde{f_n}(x)$ is a monic polynomial in x and one sees that $\widetilde{f_n}(x)$ satisfies the recurrence

$$\widetilde{f_n}(x) = x\widetilde{f_{n-1}}(x) - \frac{(n^2-1)^2}{4(4n^2-1)}\widetilde{f_{n-2}}(x) \quad \text{for } n \geq 2.$$

Since $(n^2-1)^2/4(4n^2-1) > 0$ for $n \geq 2$, this says that $\{\widetilde{f_n}(x)\}_{n=0}^{\infty}$ is a positive-definite OPS. Hence $\widetilde{f_n}(x)$ has the zeros which are all real and simple.

Therefore, we conclude that each dual polytope of the Stasheff polytope has the property ($\sharp$).

Example 7 (root polytope of type A) Let $\mathcal{P}_n = \mathrm{conv}(\{\pm\mathbf{e}_1, \ldots, \pm\mathbf{e}_n\} \cup \{\pm(\mathbf{e}_i + \cdots + \mathbf{e}_j) : 1 \leq i < j \leq n\})$. Note that this is a convex hull of the positive roots of type A Weyl group and this is the *root polytope of type A* of dimension n. Then it is known by [1] that $\delta(\mathcal{P}_n) = \left(\binom{n}{0}^2, \binom{n}{1}^2, \ldots, \binom{n}{n}^2\right)$, i.e.,

$$f_n(x) = \sum_{k=0}^{n}\binom{n}{k}^2\binom{x+n-k}{n}.$$

Note that the leading coefficient of $f_n(x)$ is equal to $\sum_{k=0}^{n}\binom{n}{k}^2/n! = \binom{2n}{n}/n!$.

Now one can check that $f_n(x)$ satisfies (1) with $M_n = (2n-1)/n^2$, that is,

$$f_n(x) = \frac{2n-1}{n^2}(2x+1)f_{n-1}(x) + \frac{(n-1)^2}{n^2}f_{n-2}(x) \quad \text{for } n \geq 2.$$

Let

$$\widetilde{f_n}(x) = \frac{n! \cdot f_n\left(\sqrt{-1}x - \frac{1}{2}\right)}{\sqrt{-1}^n \binom{2n}{n}}.$$

Then $\widetilde{f_n}(x)$ is a monic polynomial in x and one sees that $\widetilde{f_n}(x)$ satisfies the recurrence

$$\widetilde{f_n}(x) = x\widetilde{f_{n-1}}(x) - \frac{(n-1)^4}{4(2n-1)(2n-3)}\widetilde{f_{n-2}}(x) \quad \text{for } n \geq 2.$$

Since $(n-1)^4/4(2n-1)(2n-3) > 0$ for $n \geq 2$, this says that $\{\widetilde{f_n}(x)\}_{n=0}^{\infty}$ is a positive-definite OPS. Hence $\widetilde{f_n}(x)$ has the zeros which are all real and simple.

Therefore, we conclude that each root polytope of type A has the property ($\sharp$).

Example 8 (root polytope of type C) Let $\mathcal{P}_n = \mathrm{conv}(\{\pm(\mathbf{e}_i + \cdots + \mathbf{e}_{j-1}) : 1 \le i < j \le n\} \cup \{\pm(2(\mathbf{e}_i + \cdots + \mathbf{e}_{n-1}) + \mathbf{e}_n) : 1 \le i \le n-1\})$. Note that this is a convex hull of the positive roots of type C Weyl group and this is the *root polytope of type C* of dimension n. Then it is also known by [1] that $\delta(\mathcal{P}_n) = (\binom{2n}{0}, \binom{2n}{2}, \ldots, \binom{2n}{2n})$, i.e.,

$$f_n(x) = \sum_{k=0}^{n} \binom{2n}{2k} \binom{x+n-k}{n}.$$

Note that the leading coefficient of $f_n(x)$ is equal to $\sum_{k=0}^{n} \binom{2n}{2k}/n! = 2^{2n-1}/n!$.

Now one can check that $f_n(x)$ satisfies (1) with $M_n = 2/n$, that is,

$$f_n(x) = \frac{2}{n}(2x+1)f_{n-1}(x) + \frac{n-2}{n} f_{n-2}(x) \quad \text{for} \quad n \ge 2.$$

Let

$$\widetilde{f_n}(x) = \frac{n! \cdot f_n\left(\sqrt{-1}x - \frac{1}{2}\right)}{\sqrt{-1}^n 2^{2n-1}}.$$

Then $\widetilde{f_n}(x)$ is a monic polynomial in x and one sees that $\widetilde{f_n}(x)$ satisfies the recurrence

$$\widetilde{f_n}(x) = x\widetilde{f_{n-1}}(x) - \frac{(n-1)(n-2)}{16} \widetilde{f_{n-2}}(x) \quad \text{for} \quad n \ge 2.$$

Since $(n-1)(n-2)/16$ is 0 if $n = 2$, this is not an OPS.

We notice that since $f_2(x) = (2x+1)^2$, $f_n(x)$ is divisible by $(2x+1)$ for $n \ge 1$ by the above recurrence. Thus, when we let $g_n(x) = f_{n+1}(x)/(2x+1)$ for $n \ge 1$ and $g_0(x) = 1$, it is easy to see that

$$g_n(x) = \frac{1}{n}(2x+1)g_{n-1}(x) + \frac{n-1}{n} g_{n-2}(x) \quad \text{for} \quad n \ge 2.$$

This is nothing but the recurrence in Example 5. Therefore, we conclude that each root polytope of type C has the property ($\sharp$).

Remark 9 In the above four examples, each of the Ehrhart polynomials satisfies the recurrence (1) with some certain M_n. Then each M_n is actually a nonincreasing rational function on n with $0 < M_n \le 1$ for $n \ge 2$. We also notice that the above M_n's take four distinct values $1/2, 5/8, 3/4$ and 1 when $n = 2$.

Remark 10 Some of the above examples can be written as a hypergeometric function. For example,

$$\sum_{k=0}^{n} \binom{n}{k} \binom{x+n-k}{n} = {}_2F_1(-n, -x; 1; 2),$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{x+n-k}{n} = {}_3F_2(-n, n+1, -x; 1, 1; 1).$$

These are related to the *Hahn polynomial*, which is a hypergeometric orthogonal polynomial. Consult, e.g., [8, Chapter V-3].

4 Result

Finally, we discuss the existence of the other examples except for the four examples appearing in the previous section.

We consider M_n appearing in the recurrence (1). In particular, we notice the case of $n = 2$, i.e., M_2.

Here we recall the following well-known result.

Proposition 11 (cf. [16, Section 5]) *There are 16 reflexive polytopes of dimension 2 up to unimodular equivalence. In particular, there are 7 Ehrhart polynomials of reflexive polytopes of dimension 2, which are*

$$ax^2 + ax + 1, \quad a = \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4, \frac{9}{2}.$$

From this proposition, M_n appearing in (1) must be equal to one of

$$\frac{3}{8}, \frac{4}{8}, \frac{5}{8}, \frac{6}{8}, \frac{7}{8}, \frac{8}{8}, \frac{9}{8}$$

when $n = 2$.

On the one hand, as mentioned in Remark 9, we know the examples of reflexive polytopes in the case where M_2 is equal to $4/8, 5/8, 6/8$ or $8/8$.

On the other hand, when $M_2 = 9/8$, the corresponding Ehrhart polynomial of reflexive polytope of dimension 2 is $9/2x^2+9/2x+1 = (3x+1)(3x+2)/2$. Obviously, the zeros of this polynomial do not have the same real part $-1/2$.

Hence it is natural to think of the case where M_2 is equal to $3/8$ or $7/8$. The following is the main theorem of this draft, which gives one small partial answer for Problem 2.

Theorem 12 (a) *There exists a sequence of the Ehrhart polynomials of reflexive polytopes $\{i(\mathcal{P}_n, x)\}_{n=0}^{\infty}$ satisfying the three-terms recurrence* (1) *with certain $\{M_n\}_{n=2}^{\infty}$, where M_2 is one of $\{4/8, 5/8, 6/8, 8/8\}$.*
(b) *On the contrary, there exists no sequence of the Ehrhart polynomials of reflexive polytopes $\{i(\mathcal{P}_n, x)\}_{n=0}^{\infty}$ satisfying the three-terms recurrence* (1) *if we assume that M_n is a monotone decreasing rational function on n and M_2 is one of $\{3/8, 7/8\}$.*

References

[1] F. Ardila, M. Beck, S. Hoşten, J. Pfeifle and K. Seashore, Root polytopes and growth series of root lattices, *SIAM J. Discrete Math.* **25** (2011), 360–378.

[2] C. A. Athanasiadis, On a refinement of the generalized Catalan numbers for Weyl groups, *Trans. Amer. Math. Soc.* **357** (2005), 179–196.

[3] V. Batyrev, Dual polyhedra and mirror symmetry for Calabi–Yau hypersurfaces in toric varieties, *J. Algebraic Geom.* **3** (1994), 493–535.

[4] M. Beck, J. A. De Loera, M. Develin, J. Pfeifle and R. P. Stanley, Coefficients and roots of Ehrhart polynomials, *Contemp. Math.* **374** (2005), 15–36.

[5] M. Beck and S. Robins, "Computing the Continuous Discretely," Undergraduate Texts in Mathematics, Springer, 2007.

[6] C. Bey, M. Henk and J. M. Wills, Notes on the roots of Ehrhart polynomials, *Discrete Comput. Geom.* **38** (2007), 81–98.

[7] B. Braun, Norm bounds for Ehrhart polynomial roots, *Discrete Comput. Geom.* **39** (2008), 191–193.

[8] T. S. Chihara, "An Introduction to Orthogonal Polynomials," Gordon and Breach, New York, 1978.

[9] E. Ehrhart, "Polynômes Arithmétiques et Méthode des Polyèdres en Combinatoire," Birkhäuser, Boston/Basel/Stuttgart, 1977.

[10] G. Hegedüs and A. M. Kasprzyk, Roots of Ehrhart polynomials of smooth Fano polytopes, *Discrete Comput. Geom.* **46** (2011) 488–499.

[11] T. Hibi, "Algebraic Combinatorics on Convex Polytopes," Carslaw Publications, Glebe, N.S.W., Australia, 1992.

[12] T. Hibi, Dual polytopes of rational convex polytopes, *Combinatorica* **12** (1992), 237–240.

[13] A. Higashitani, Counterexamples of the Conjecture on Roots of Ehrhart Polynomials, *Discrete Comput. Geom.* **47** (2012) 618–623.

[14] T. Matsui, A. Higashitani, Y. Nagazawa, H. Ohsugi and T. Hibi, Roots of Ehrhart polynomials arising from graphs, *J. Algebr. Comb.* **34** (2011), no. 4 721–749.

[15] H. Ohsugi and K. Shibata, Smooth Fano polytopes whose Ehrhart polynomial has a root with large real part, *Discrete Comput. Geom.* **47** (2012) 624–628.

[16] B. Poonen and F. Rodriguez-Villegas, Lattice Polygons and the Number 12, *Amer. Math. Monthly* **107** (2000), 238–250.

[17] R. P. Stanley, Decompositions of rational convex polytopes, *Annals of Discrete Math.* **6** (1980), 333–342.

[18] R. P. Stanley, "Enumerative Combinatorics, Volume 1," Wadsworth & Brooks/Cole, Monterey, Calif., 1986.